27/9/23

MATH2050A Tutorial

__Recall__: Nested Interval Thm: Suppose $\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ is a sequence of nested closed bounded intervals, then

$$\emptyset \neq \bigcap_{n=1}^{\infty} I_n = I_1 \cap I_2 \cap I_3 \cap \cdots$$

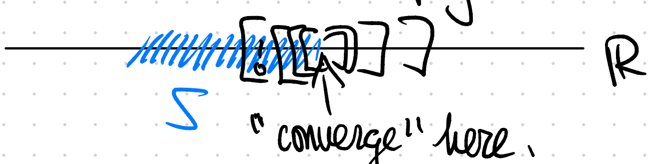
and if $\inf\{b_n - a_n\} = 0$, then $\exists!\, z \in \mathbb{R}$ s.t. $z \in \bigcap_{n=1}^{\infty} I_n$.

__Thm__: NIT + AP. $\Leftrightarrow$ completeness Axiom.

__Pf__: $\Leftarrow$: done in lecture when NIT was proved.

(existence of $\inf\{b_n : n \in \mathbb{N}\}$, $\sup\{a_n : n \in \mathbb{N}\}$ requires completeness axiom).

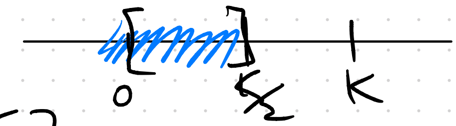
$\Rightarrow$: Let $S \subseteq \mathbb{R}$ nonempty and bounded from above, by $K \in \mathbb{R}$.



Since S is non-empty, $\exists\, s_0 \in S$.
Without loss of generality (WLOG),
take $s_0 = 0$.

$s_0 \in S \subseteq \mathbb{R}$, take the set

$S - s_0 = \{s - s_0 : s \in S\}.$

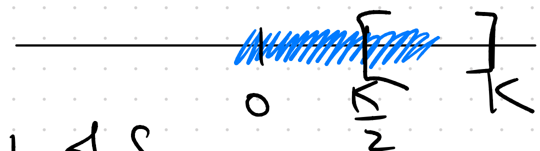


Take $I_1 = [0, K]$.

If $\frac{K}{2}$ is an upper bound of S, take $I_2 = [0, \frac{K}{2}]$

o/w take $I_2 = [\frac{K}{2}, K]$. ($\frac{K}{2}$ is in S b/c it is not an u.b. of S and $0 \le \frac{K}{2}$).

Iterate this to obtain a sequence of nested intervals



$I_n = [a_n, b_n]$ s.t.

① b_n is an u.b. of S

② $\exists s_n \in S$ s.t. $s_n \in [a_n, b_n]$.

③ length of $I_n = \frac{K}{2^n}$ " $\longrightarrow 0$ as $n \to \infty$." AP.

By NIT, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ and $\overset{\text{by ③}}{\exists!}\, z \in \mathbb{R}$ s.t. $z \in \bigcap_{n=1}^{\infty} I_n$.

NIT + AP $\Rightarrow$ Completeness axiom

WTS $z = \sup S$.

Note that since $z \in \bigcap_{n=1}^{\infty} I_n$, $z \in [a_n, b_n]$ for all n.

$\Rightarrow \quad a_n \le z \le b_n$. for all n.

z is an u.b. of S. Suppose $\exists s \in S$ s.t. $z < s$.

Since $z \in \bigcap_{n=1}^{\infty} I_n$ is unique, $s \notin \bigcap_{n=1}^{\infty} I_n$. In particular, $\exists m \in \mathbb{N}$ s.t.

$s \notin I_m = [a_m, b_m]$.

1) $s < a_m$. $\Leftarrow$ $s < a_m \le z$ contradicts $z < s$.

2) $b_m < s$. $\Leftarrow$ $z \le b_m < s$, which contradicts b_m is an upper bound of S.

So z is an u.b. of S.

Let v be an u.b. of S. Suppose $v < z$. Again since $v \notin \bigcap_{n=1}^{\infty} I_n$,

$\exists m \in \mathbb{N}$ s.t. $v \notin I_m$. Then either

$v < a_m$, $b_m < v$.

contradiction!

Q1: Let $\{I_n = [a_n, b_n]\}_{n=1}^{\infty}$ be a sequence of nested closed bdd. intervals
Let $\xi = \sup\{a_n : n \in \mathbb{N}\}$, $\eta = \inf\{b_n : n \in \mathbb{N}\}$.
Show that $\eta \in \bigcap_{n=1}^{\infty} I_n$, and $[\xi, \eta] = \bigcap_{n=1}^{\infty} I_n$.

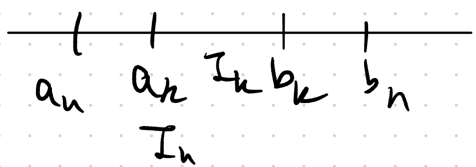
Pf: WTS $\eta \in \bigcap_{n=1}^{\infty} I_n$. Need to show $\eta \in I_n$ for each n.

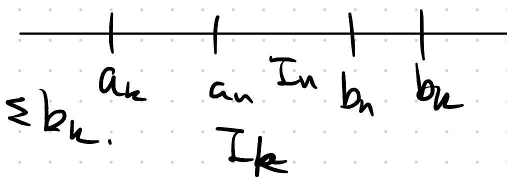
Clearly (by def'n), $\eta \leq b_n$ for all n.

Remains to show $a_n \leq \eta$ for all n.

Fix n. We will show a_n is a lower bdd for the set $\{b_k : k \in \mathbb{N}\}$.

Case 1: if $n \leq k$, then $I_n \supseteq I_k$. we have $a_n \leq a_k \leq b_k \leq b_n$.





Case 2: $k < n$. Then $I_k \supseteq I_n$ i.e. $a_k \leq a_n \leq b_n \leq b_k$.

Either way. $a_n \leq b_k$ for all k.

By def'n of inf, $a_n \leq \eta \leq b_k$ for all k, n.

So $\eta \in [a_n, b_n]$ for all n. $\Rightarrow \eta \in \bigcap_{n=1}^{\infty} I_n$.

similarly $\Rightarrow \zeta \in \bigcap_{n=1}^{\infty} I_n$.

So $[\zeta, \eta] \subseteq \bigcap_{n=1}^{\infty} I_n$.

So it remains to show: $\bigcap_{n=1}^{\infty} I_n \subseteq [\zeta, \eta]$.

Let $z \in \bigcap_{n=1}^{\infty} I_n$. Then $z \in [a_n, b_n]$ for all n

$a_n \leq z \leq b_n$. $\Rightarrow a_n \leq \zeta \leq z \leq \eta \leq b_n$.

def'n of sup, inf

$\Rightarrow z \in [\zeta, \eta]$. $\Rightarrow \bigcap_{n=1}^{\infty} I_n = [\zeta, \eta]$. ✓

① To show sets $A = B$, show $A \subseteq B$, $B \subseteq A$.

② $\{A_n\}_{n=1}^{\infty}$. $z \in \bigcap_{n=1}^{\infty} A_n \Rightarrow z \in A_n$ for each n

$z \in \bigcup_{n=1}^{\infty} A_n \Rightarrow z \in A_n$ for at least one n.

(3) Think about what it means if

$$z \notin \bigcap_{n=1}^{\infty} A_n$$

$$z \notin \bigcup_{n=1}^{\infty} A_n$$

$$z \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$$z \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$